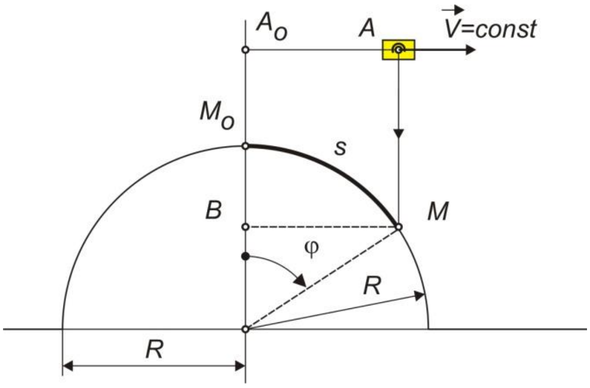
Zadatak 9.

Svetlosni izvor A ispusta koncentrisan mlaz svetlosti vertikalno naniže iz početnog položaja A_o i kreće se po horizontali konstantnom brzinom V_o. Mlaz na svom putu nailazi na polukružni fluorescentni zastor obrazujući na njemu svetlu mrlju M koja se kreće po zastoru. Odrediti lučnu koordinatu tačke M, njenu brzinu i ubrzanje (normalno i tangencijalno)

a) U proizvoljnom trenutku vremena
b) U položaju tačke M određene uglom $\varphi_1 = \frac{\pi}{4}$.



Rešenje:

$$s = R\varphi$$

$$\overline{A_oA} = \overline{BM} = V_ot = Rsin\varphi$$

$$\varphi = arcsin\left(\frac{V_ot}{R}\right)$$

$$(arcsinx)' = \frac{1}{\sqrt{1-x^2}}$$

$$\left(\frac{C}{v}\right)' = -\frac{C\,v'}{v^2}$$

$$\left(\sqrt{x}\right)' = -\frac{1}{2\sqrt{x}} \cdot x'$$

Prirodna koordinata tačke M

$$s = R\varphi = Rarcsin\left(\frac{V_ot}{R}\right)$$

Brzina tačke M

$$V = \frac{ds}{dt} = \frac{d}{dt}\left[Rarcsin\left(\frac{V_ot}{R}\right)\ \right] = R\frac{1}{\sqrt{1-\left(\frac{V_ot}{R}\right)^2}} \cdot \frac{V_o}{R} = \frac{V_oR}{\sqrt{R^2-{V_o}^2t^2}}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}\left[\frac{V_oR}{\sqrt{R^2-{V_o}^2t^2}}\right] = -\frac{V_oR\cdot(-2V_o^2t)}{2\sqrt{R^2-{V_o}^2t^2}\left(R^2-{V_o}^2t^2\right)} = \frac{V_o^3Rt}{\sqrt{\left(R^2-{V_o}^2t^2\right)^3}}$$

$$a_N = \frac{V^2}{R} = \frac{1}{R}\cdot\frac{{V_o}^2R^2}{R^2-{V_o}^2t^2} = \frac{{V_o}^2R}{R^2-{V_o}^2t^2}$$

$$a = \sqrt{a_T^2 + a_N^2} = \sqrt{\left(\frac{V_o^3Rt}{\sqrt{\left(R^2-{V_o}^2t^2\right)^3}}\right)^2 + \left(\frac{{V_o}^2R}{R^2-{V_o}^2t^2}\right)^2} = \frac{V_o^2R^2}{\sqrt{\left(R^2-{V_o}^2t^2\right)^3}}$$

$$V_o t = R sin\varphi = R sin\frac{\pi}{4} = R\frac{\sqrt{2}}{2} \rightarrow t_1 = \frac{R}{V_o}\frac{\sqrt{2}}{2}$$

$$a_{T1} = \frac{V_o^3 R\frac{R}{V_o}\frac{\sqrt{2}}{2}}{\sqrt{\left(R^2 - {V_o}^2\left(\frac{R}{V_o}\frac{\sqrt{2}}{2}\right)^2\right)^3}} = \frac{V_o^2 R^2 \frac{\sqrt{2}}{2}}{\sqrt{\left(R^2 - {V_o}^2\frac{R^2}{{V_o}^2}\cdot\frac{1}{2}\right)^3}} = \frac{V_o^2 R^2\sqrt{2}}{2\sqrt{\left[R^2\left(\frac{2-1}{2}\right)\right]^3}} = \frac{V_o^2 R^2\sqrt{2}}{2\sqrt{\left(\frac{R^2}{2}\right)^3}}$$

$$a_{T1} = \frac{V_o^2 R^2\sqrt{2}\cdot\sqrt{8}}{2R^3} = \frac{2V_o^2}{R}$$

$$a_{N1} = \frac{{V_o}^2 R}{R^2 - {V_o}^2\left(\frac{R}{V_o}\frac{\sqrt{2}}{2}\right)^2} = \frac{{V_o}^2 R}{R^2 - {V_o}^2\frac{R^2}{{V_o}^2}\cdot\frac{1}{2}} = \frac{2V_o^2}{R}$$

$$a = \sqrt{a_T^2 + a_N^2} = \sqrt{\left(\frac{2V_o^2}{R}\right)^2 + \left(\frac{2V_o^2}{R}\right)^2} = \frac{2V_o^2}{R}\sqrt{2}$$

Zadatak 10.

Voz, koji se kreće jednakousporeno u krivini poluprečnika R=800m, pređe put od s=800m. Početna brzina V_o=54 km/h, a krajnja V=18 km/h.

Odrediti totalno ubrzanje voza na početku i na kraju krivine, kao i vreme koje protekne dok se kreće na tom putu.

$$V_o = 54\ \frac{km}{h}\frac{1}{3.6} = 15\frac{m}{s} \quad V = 18\ \frac{km}{h}\frac{1}{3.6} = 5\frac{m}{s}$$

Rešenje:

$$\frac{dV}{dt} = \frac{d^2 s}{dt^2} = \ddot{s} = -a_T = const$$

$$\int dV = -a_T \int dt$$

$V = -a_T t + C_1$ za početak krivine $V = V_o \rightarrow C_1 = V_o = 15$

$$V = V_o - a_T t$$

$$s = \int (V_o - a_T t)dt = \int V_o dt - \int a_T t dt = V_o t - \frac{a_T t^2}{2} + C_2$$

za početak krivine $s_o = 0 \rightarrow C_2 = 0$

$$s = V_o t - \frac{a_T t^2}{2}$$

$$a_T = \frac{V_o - V}{t} \rightarrow t = \frac{V_o - V}{a_T}$$

$$s = V_o t - \frac{a_T t^2}{2} = V_o\frac{V_o - V}{a_T} - \frac{a_T\left(\frac{V_o - V}{a_T}\right)^2}{2} = \frac{{V_o}^2 - V_o V}{a_T} - \frac{{V_o}^2 - 2V_o V + V^2}{2a_T} = \frac{{V_o}^2 - V^2}{2a_T}$$

$$s = \frac{{V_o}^2 - V^2}{2a_T} = \frac{225-25}{2a_T} = 800\,m \rightarrow a_T = \frac{200}{1600} = \frac{1}{8} m/s^2$$

$$t = \frac{V_o - V}{a_T} = \frac{15-5}{\frac{1}{8}} = 80\,s$$

U početnom trenutku

$$a_N = \frac{V_o^2}{R} = \frac{225}{800} = \frac{9}{32} m/s^2$$

$$a_T = \frac{1}{8} m/s^2$$

$$a = \sqrt{a_T^2 + a_N^2} = \sqrt{\left(\frac{1}{8}\right)^2 + \left(\frac{9}{32}\right)^2} = \frac{\sqrt{97}}{32} = 0.3077 m/s^2$$

U krajnjem položaju

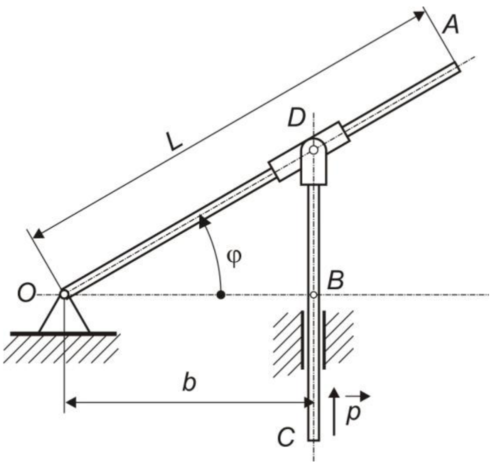
$$a_N = \frac{V^2}{R} = \frac{25}{800} = \frac{1}{32} m/s^2$$

$$a = \sqrt{a_T^2 + a_N^2} = \sqrt{\left(\frac{1}{8}\right)^2 + \left(\frac{1}{32}\right)^2} = \frac{\sqrt{17}}{32} = 0.1288\, m/s^2$$

Zadatak 11.

Kulisni mehanizam se sastoji od poluge $\overline{OA}$, koja može da se obrće u ravni crteža oko ose O upravne na ravan crteža, i vertikalnog podizača $\overline{CD}$, koji je preko klizača D povezan sa polugom $\overline{OA}$. Podizač se kreće vertikalno na više kroz vođicu B konstantnom brzinom p. Ako je u početnom trenutku t_o=0, poluga OA bila horizontalna, odrediti brzinu i ubrzanje tačke A u proizvoljnom trenutku t i u trenutku kada je poluga $\overline{OA}$ pod uglom od $\varphi_1 = \frac{\pi}{4}$ u odnosu na horizontalnu ravan koristeći

a) Dekartov koordinatni sistem
b) Prirodni koordinatni sistem .



Rešenje:

Iz opisa kulisnog mehanizma i sa slike je jasno da poluga $\overline{OA}$ vrši okretanje oko ose O, odnosno da se tačka A kreće po kružnici poluprečnika L. Kretanje tačke A najjednostavnije je pratiti praćenjem promene ugla φ. Tačka D se kreće pravolinijski konstantnom brzinom *p=const* na više.

$$dy_D = pdt \rightarrow \int_o^D dy_d = \int_{t=o}^t pdt = p\int_{t=o}^t dt$$

$y_D = pt + C_1$ kako je rečeno za t=0, AD je horizontalno $y_{Do} = 0 \rightarrow C_1 = 0$

$y_D = pt$ koordinata tačke D jednaka je duži BD, $\overline{BD} = pt$ pa je

$$tg\varphi = \frac{\overline{BD}}{\overline{OB}} = \frac{pt}{b} \rightarrow \varphi = arctg\,\frac{pt}{b}$$

$$(arctx)' = \frac{1}{1+x^2}$$

$$(cv)' = -\frac{cv'}{v^2}$$

$$\varphi = arctg\,\frac{pt}{b}$$

$$\dot{\varphi} = \frac{1}{1+\left(\frac{pt}{b}\right)^2} \cdot \frac{p}{b} = \frac{b^2}{b^2+p^2t^2} \cdot \frac{p}{b} = \frac{pb}{b^2+p^2t^2}$$

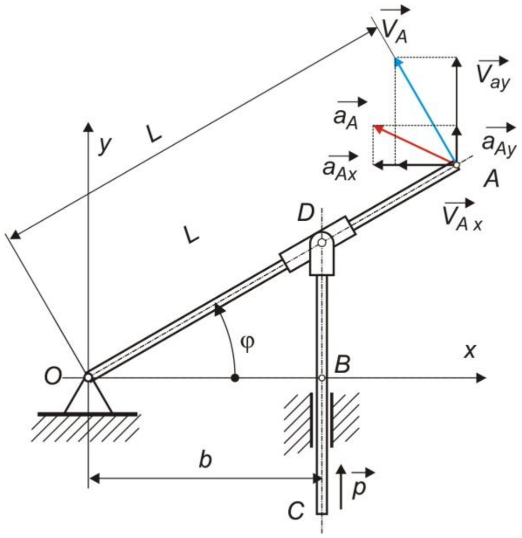
$$\ddot{\varphi} = \frac{-pb \cdot 2p^2t}{(b^2+p^2t^2)^2} = \frac{-2p^3bt}{(b^2+p^2t^2)^2}$$

a) U Dekatovim koordinatama

$x_A = L\,cos\varphi$ treba imati na umu da je $\varphi = arctg\,\frac{pt}{b}$

$y_A = L\,sin\varphi$

Kvadriranjem i sabiranjem jednačina eliminiše se f parametar u funkciji vremena pa se dobija trajektorija tačke A



$$x_A^2 + y_A^2 = L^2$$

Vidi se da je to kružnica poluprečnika L

$$\dot{x}_A = -L\,\dot{\varphi}sin\varphi$$

$$\dot{y}_A = L\,\dot{\varphi}cos\varphi$$

$$\overrightarrow{V_A} = V_{Ax} \cdot \vec{\imath} + V_{Ay} \cdot \vec{\jmath} = \dot{x}_A \cdot \vec{\imath} + \dot{y}_A \cdot \vec{\jmath}$$

$$\overrightarrow{V_A} = -L\,\dot{\varphi}sin\varphi \cdot \vec{\imath} + L\,\dot{\varphi}cos\varphi \cdot \vec{\jmath}$$

$$\overrightarrow{V_A} = -L\,\left(\frac{pb}{b^2+p^2t^2}\right) sin\varphi \cdot \vec{\imath} + L\,\left(\frac{pb}{b^2+p^2t^2}\right) cos\varphi \cdot \vec{\jmath}$$

$$\overrightarrow{V_A} = \frac{-Lpbsin\varphi}{b^2+p^2t^2} \cdot \vec{\imath} + \frac{Lpbcos\varphi}{b^2+p^2t^2} \cdot \vec{\jmath}$$

$$V_A = \sqrt{L^2\,\dot{\varphi}^2 sin^2\varphi + L^2\,\dot{\varphi}^2 cos^2\varphi} = L\,\dot{\varphi} = \frac{Lpb}{b^2+p^2t^2}$$

$$\vec{a}_A = a_{Ax} \cdot \vec{\imath} + a_{Ay} \cdot \vec{\jmath} = \ddot{x}_A \cdot \vec{\imath} + \ddot{y}_A \cdot \vec{\jmath}$$

$$\ddot{x}_A = -L\ddot{\varphi}sin\varphi - L\,\dot{\varphi}^2 cos\varphi = \frac{L2p^3btsin\varphi}{(b^2+p^2t^2)^2} - \frac{Lp^2b^2cos\varphi}{(b^2+p^2t^2)^2}$$

$$\ddot{y}_A = L\,\ddot{\varphi}cos\varphi - L\,\dot{\varphi}^2 sin\varphi = \frac{-L2p^3btcos\varphi}{(b^2+p^2t^2)^2} - \frac{Lp^2b^2sin\varphi}{(b^2+p^2t^2)^2}$$

$$\vec{a}_A = (-L\ddot{\varphi}sin\varphi - L\,\dot{\varphi}^2cos\varphi\,) \cdot \vec{\imath} + (L\,\ddot{\varphi}cos\varphi - L\,\dot{\varphi}^2 sin\varphi) \cdot \vec{\jmath}$$

$$\vec{a}_A = \left(\frac{L2p^3btsin\varphi}{(b^2+p^2t^2)^2} - \frac{Lp^2b^2cos\varphi}{(b^2+p^2t^2)^2}\right) \cdot \vec{\imath} + \left(\frac{-L2p^3btcos\varphi}{(b^2+p^2t^2)^2} - \frac{Lp^2b^2sin\varphi}{(b^2+p^2t^2)^2}\right) \cdot \vec{\jmath}$$

$a_A = \sqrt{a_{Ax}^2 + a_{Ay}^2}$

$a_A = \sqrt{(-L\ddot{\varphi}sin\varphi - L\,\dot{\varphi}^2cos\varphi\,)^2 + (L\,\ddot{\varphi}cos\varphi - L\,\dot{\varphi}^2sin\varphi)^2}$

$a_A = L\sqrt{\ddot{\varphi}^2 + \dot{\varphi}^2}$

Za ugao $\varphi_1 = \frac{\pi}{4}$

$tg\varphi = \frac{pt}{b}$

$tg\varphi_1 = tg\frac{\pi}{4} = 1 = \frac{pt}{b} \rightarrow t_1 = \frac{b}{p}$

$\dot{\varphi} = \frac{pb}{b^2+p^2\left(\frac{b}{p}\right)^2} = \frac{p}{2b}$

$\ddot{\varphi} = \frac{2p^3b\left(\frac{b}{p}\right)}{\left(b^2+p^2\left(\frac{b}{p}\right)^2\right)^2} = -\frac{p^2}{2b^2}$

$\dot{x}_A = -L\,\dot{\varphi}sin\varphi = -L\left(\frac{p}{2b}\right)sin\frac{\pi}{4} = -\frac{Lp\sqrt{2}}{4b}$

$\dot{y}_A = L\,\dot{\varphi}cos\varphi = L\left(\frac{p}{2b}\right)cos\frac{\pi}{4} = \frac{Lp\sqrt{2}}{4b}$

$V_{A1} = L\,\dot{\varphi} = \frac{Lp}{2b}$

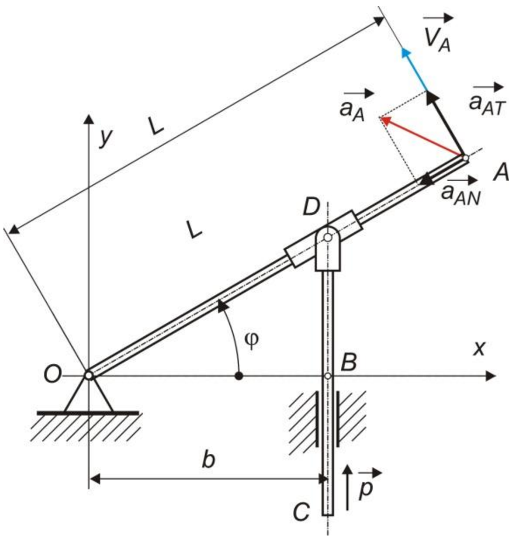
$\ddot{x}_A = -L\ddot{\varphi}sin\varphi - L\,\dot{\varphi}^2cos\varphi = -L\left(-\frac{p^2}{2b^2}\right)\frac{\sqrt{2}}{2} - L\left(\frac{p}{2b}\right)^2\frac{\sqrt{2}}{2} = \frac{Lp^2\sqrt{2}}{8b^2}$

$\ddot{y}_A = L\,\ddot{\varphi}cos\varphi - L\,\dot{\varphi}^2sin\varphi = L\left(-\frac{p^2}{2b^2}\right)\frac{\sqrt{2}}{2} - L\left(\frac{p}{2b}\right)^2\frac{\sqrt{2}}{2} = \frac{-3Lp^2\sqrt{2}}{8b^2}$

$\vec{a}_A = \ddot{x}_A \cdot \vec{\imath} + \ddot{y}_A \cdot \vec{\jmath}$

$\vec{a}_A = \frac{Lp^2\sqrt{2}}{8b^2} \cdot \vec{\imath} + \frac{-3Lp^2\sqrt{2}}{8b^2} \cdot \vec{\jmath}$

$a_A = \frac{Lp^2\sqrt{5}}{4b^2}$



b) U prirodnim koordinatama

$s_A = L \cdot \varphi$

$V_A = \frac{ds_A}{dt} = \dot{s}_A = L \cdot \dot{\varphi} = \frac{L\,pb}{b^2+p^2t^2}$

$a_{AT} = \frac{d\dot{s}_A}{dt} = \ddot{s}_A = L \cdot \ddot{\varphi} = \frac{-2L\,p^3bt}{\left(b^2+p^2t^2\right)^2}$

$$a_{AN} = \frac{V_A^2}{L} = L\dot{\varphi}^2 = \frac{L\,p^2 b^2}{\left(b^2+p^2t^2\right)^2}$$

$$a_A = \sqrt{a_{AT}^2 + a_{AN}^2} = L\sqrt{\ddot{\varphi}^2 + \dot{\varphi}^2} = \sqrt{\left(\frac{2L\,p^3bt}{(b^2+p^2t^2)^2}\right)^2 + \left(\frac{L\,p^2b^2}{(b^2+p^2t^2)^2}\right)^2}$$

$$a_A = \frac{L}{(b^2+p^2t^2)^2}\sqrt{4\,p^6b^2t^2 + p^4b^4}$$

Za ugao $\varphi_1 = \frac{\pi}{4}$

$$tg\varphi = \frac{pt}{b}$$

$$tg\varphi_1 = tg\frac{\pi}{4} = 1 = \frac{pt}{b} \rightarrow t_1 = \frac{b}{p}$$

$$\dot{\varphi} = \frac{pb}{b^2+p^2\left(\frac{b}{p}\right)^2} = \frac{p}{2b}$$

$$\ddot{\varphi} = \frac{2p^3b\left(\frac{b}{p}\right)}{\left(b^2+p^2\left(\frac{b}{p}\right)^2\right)^2} = -\frac{p^2}{2b^2}$$

$$V_A = L\cdot\dot{\varphi} = L\frac{p}{2b}$$

$$a_{A1T} = L\cdot\ddot{\varphi} = -L\frac{p^2}{2b^2}$$

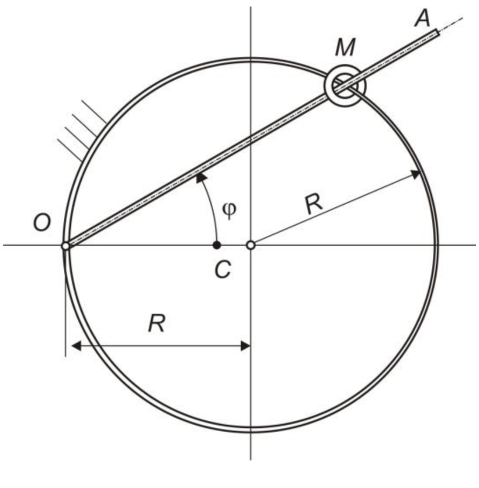
$$a_{A1N} = L\dot{\varphi}^2 = L\frac{p^2}{4b^2}$$

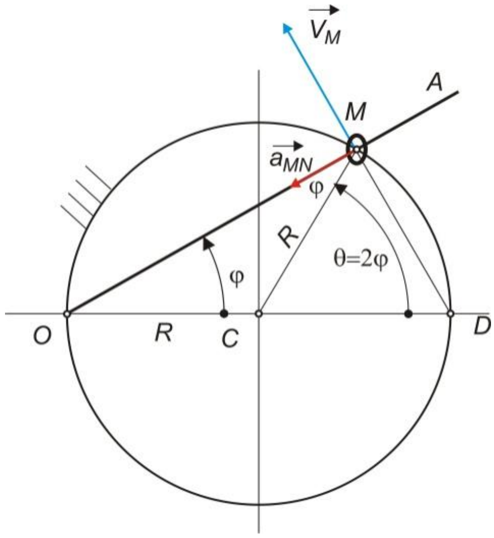
$$a_A = L\frac{p^2\sqrt{5}}{4b^2}$$

Zadatak 12.

Na nepomičnom kružnom obruču, poluprečnika R, nalazi se prsten M kroz koji prolazi poluga $\overline{OA}$. Poluga se okreće ravnomerno, u ravni crteža, oko tačke O koja se nalazi na obodu obruča. U početnom trenutku poluga OA je horizontalna, a za vreme od 5 sekundi opiše ugao od $\varphi_1 = \frac{\pi}{2}$. Odrediti brzinu i ubrzanje tačke M u proizvoljnom trenutku koristeći

a) Prirodni koordinatni sistem
b) Dekartov koordinatni sistem





Prsten je geometrijskom vezom prinuđen da se kreće kružno po prstenu oko centra C.

Trougao OCM je jednakokraki pa je treći ugao 180 -2φ, a suplementan θ=2φ

Poluga se okreće ravnomerno pa je

$$\frac{d\varphi}{dt} = \dot{\varphi} = \dot{\varphi_{SR}} = \frac{\varphi_1 - \varphi_0}{t_1 - t_0} = \frac{\frac{\pi}{2}}{5} = \frac{\pi}{10}$$

$$\theta = 2\varphi \rightarrow \frac{d\theta}{dt} = \frac{d}{dt}(2\varphi) = 2\frac{d\varphi}{dt}$$

$$\frac{d\theta}{dt} = \dot{\theta} = 2\frac{d\varphi}{dt} = \frac{\pi}{5}$$

Prirodna koordinata tačke M pređeni put od D do M

$$\theta = \int \dot{\theta}\, dt = \int \frac{\pi}{5} dt = \frac{\pi}{5} t + C_1$$

Kako se ugao menja od horizontale a od te ose se i meri $t_0 = 0, \theta_0 = 0 \rightarrow C_1 = 0$

$$s = \int R\theta dt = \int R\frac{\pi}{5} dt = R\frac{\pi}{5}\int dt = R\frac{\pi}{5} t + C_2$$

Kako se put - koordinata menja od horizontale $t_0 = 0, s_0 = 0 \rightarrow C_2 = 0$

$$s = R\frac{\pi}{5} t$$

$$\theta = \frac{\pi}{5} t \qquad s = R\frac{\pi}{5} t$$

$$\dot{\theta} = \frac{\pi}{5} \qquad \dot{s} = R\frac{\pi}{5}$$

$$\ddot{\theta} = 0 \qquad \ddot{s} = 0$$

Brzina prstena M

$$V_M = \frac{ds}{dt} = \dot{s} = R\dot{\theta} = R\frac{\pi}{5}$$

Ubrzanje prstena M

$$a_{MT} = \frac{d\dot{s}_A}{dt} = \ddot{s_A} = R \cdot \ddot{\theta} = 0$$

$$a_{MN} = \frac{V_M^2}{R} = R\dot{\theta}^2 = R\left(\frac{\pi}{5}\right)^2 = \frac{R\pi^2}{25}$$

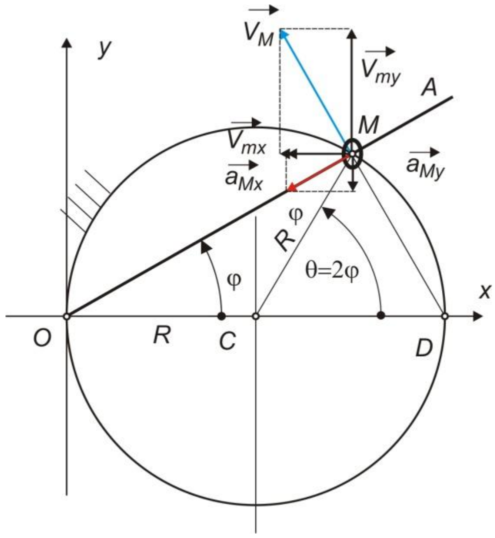
$$a_M = \sqrt{a_{MT}^2 + a_{MN}^2} = \sqrt{0 + \left(\frac{R\pi^2}{25}\right)^2} = \frac{R\pi^2}{25}$$

Prirodna koordinata tačke M pređeni put od D do M

$$\theta = \int \dot{\theta}\, dt = \int \frac{\pi}{5} dt = \frac{\pi}{5} t + C_1$$

Kako se ugao menja od horizontale a od te ose se i meri $t_0 = 0, \theta_0 = 0 \rightarrow C_1 = 0$

$$\theta = \frac{\pi}{5} t$$



$$\dot{\theta} = \frac{\pi}{5}$$

$$\ddot{\theta} = 0$$

$$x = R + R\, cos\theta$$

$$y = R\, sin\theta$$

$$\dot{x} = -R\, \dot{\theta} sin\theta = -R \frac{\pi}{5} sin\left(\frac{\pi}{5} t\right)$$

$$\dot{y} = R\dot{\theta} cos\theta = R \frac{\pi}{5}\, cos\left(\frac{\pi}{5} t\right)$$

$$\ddot{x} = -R\ddot{\theta} sin\theta - R\, \dot{\theta}^2 cos\theta = 0 - R\, \dot{\theta}^2 cos\theta = -R\left(\frac{\pi}{5}\right)^2 cos\left(\frac{\pi}{5} t\right)$$

$$\ddot{y} = R\ddot{\theta} cos\theta - R\, \dot{\theta}^2 sin\theta = 0 - R\, \dot{\theta}^2 sin\theta = -R\left(\frac{\pi}{5}\right)^2 sin\left(\frac{\pi}{5} t\right)$$

$$\overrightarrow{V_M} = V_{Mx} \cdot \vec{\imath} + V_{My} \cdot \vec{\jmath} = \dot{x_M} \cdot \vec{\imath} + \dot{y_M} \cdot \vec{\jmath}$$

$$\overrightarrow{V_M} = -R \frac{\pi}{5} sin\left(\frac{\pi}{5} t\right) \cdot \vec{\imath} + R \frac{\pi}{5}\, cos\left(\frac{\pi}{5} t\right) \cdot \vec{\jmath}$$

$$V_A = \sqrt{\left(-R \frac{\pi}{5} sin\left(\frac{\pi}{5} t\right)\right)^2 + \left(R \frac{\pi}{5}\, cos\left(\frac{\pi}{5} t\right)\right)^2} = R \frac{\pi}{5}$$

$$\vec{a}_M = \ddot{x}_M \cdot \vec{\imath} + \ddot{y}_M \cdot \vec{\jmath} == -R\left(\frac{\pi}{5}\right)^2 cos\left(\frac{\pi}{5} t\right) \cdot \vec{\imath} - R\left(\frac{\pi}{5}\right)^2 sin\left(\frac{\pi}{5} t\right) \cdot \vec{\jmath}$$

$$a_M = \sqrt{a_{Mx}^2 + a_{My}^2} \quad a_A = \sqrt{\left(-R\left(\frac{\pi}{5}\right)^2 cos\left(\frac{\pi}{5} t\right)\right)^2 + \left(-R\left(\frac{\pi}{5}\right)^2 sin\left(\frac{\pi}{5} t\right)\right)^2}$$

$$a_M = R \frac{\pi^2}{25}$$