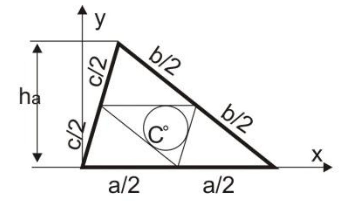
TEŽIŠTE I PAPUS GULDENOVE TEOREME

Težište tela je tačka kroz koju uvek prolazi napadna linija sile težine bez obzira na položaj tela. Težina je rezultanta svih sila težine pojedinih delova materijalnog tela.

Postoji više metoda za određivanje težišta: analitički, grafički, eksperimentalne metode

Određivanje položaja težišta homogenih ravnih figura i linija.

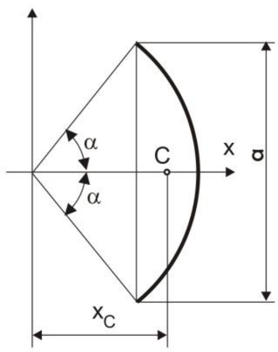


Obim trougla

$$y_C = \frac{h_a}{2}\,\frac{b+c}{a+b+c}$$

2π rad = 360°

$$1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.2957795^\circ = 57^\circ 17' 44{,}8''$$



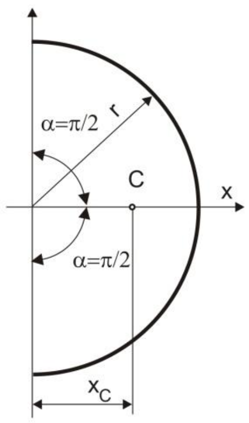
Kružni luk

$$x_C = \frac{ar}{l} = \frac{r \sin\alpha}{\alpha}$$

$a = 2r\sin\alpha$ - tetiva

$l = 2r\alpha$ dužina kružnog luka

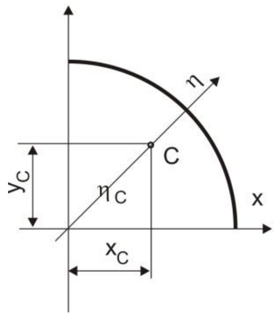
2α - centralni ugao u radijanima



Poluobim kruga

$$x_C = \frac{2r}{\pi}$$

$l = r\pi$ dužina kružnog luka



Kvadrant kružne linije

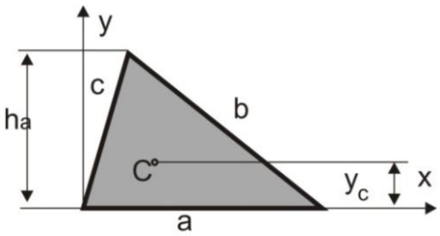
$\alpha = \pi/4$

$2\alpha = \pi/2$

$$\eta = \frac{2r\sqrt{2}}{\pi}$$

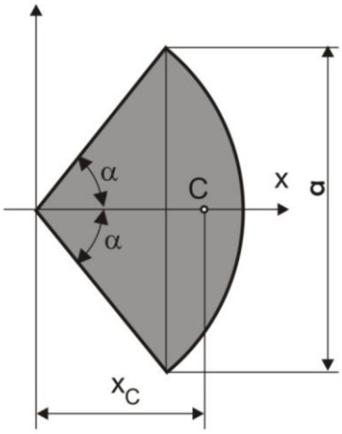
$$x_C = y_c = \frac{2r}{\pi}$$

Koordinate težišta homogenih površina



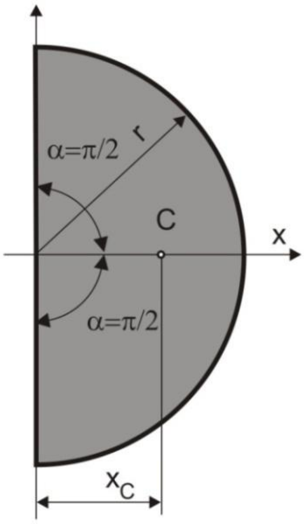
$A = \frac{a \cdot h}{2}$

$y_C = \frac{h}{3}$



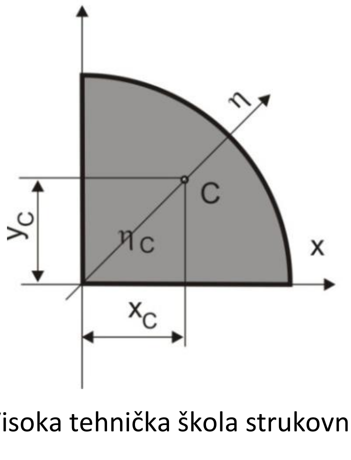
$x_C = \frac{4\ r\ sin\alpha}{3\alpha}$

$A = r^2\alpha$



$x_C = \frac{4\ r}{3\pi}$

$A = \frac{r^2\pi}{2}$



$\eta = \frac{4\ r\sqrt{2}}{3\pi}$

$x_C = y_c = \frac{4\ r}{3\pi}$

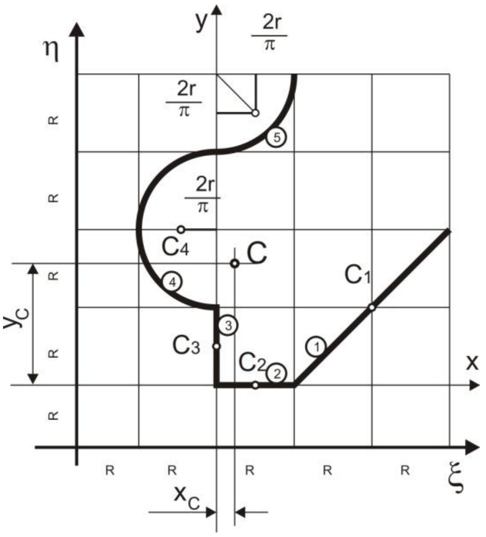
$A = \frac{r^2\pi}{2}$

Papus Guldinova teoreme

$A_x = \varphi L y_c = 2\pi L y_c \frac{\alpha^\circ}{360^\circ}$ $\quad V_x = \varphi A y_c = 2\pi A y_c \frac{\alpha^\circ}{360^\circ}$

Gde je φ u radijanima, a α u stepenima

Primer 3.1



Za prikazanu krivu liniju kada je R=10 cm.

1. Odrediti koordinate težišta krive linije za date koordinatne ose xOy
2. Obrtnu površinu koja nastaje obrtanjem oko ξ ose za pun krug.
3. Obrtnu površinu koja nastaje obrtanjem oko η ose za 180^{o}.

$L_1=2R\sqrt{2}$	$x_1=2R$,	$y_1=R$
$L_2=R$	$x_2=0{,}5R$,	$y_2=0$
$L_3=R$	$x_3=0$,	$y_3=0{,}5R$
$L_4=R\pi$	$x_4=-\frac{2R}{\pi}$,	$y_4=2R$
$L_5=\frac{R\pi}{2}$	$x_5=\frac{2R}{\pi}$,	$y_5=4R-\frac{2R}{\pi}$

$$L=\sum L_i=L_1+L_2+L_3+L_4+L_5=2R\sqrt{2}+R+R+R\pi+\frac{R\pi}{2}=9.538R=95.38cm$$

$$x_C=\frac{L_1\cdot x_1+L_2\cdot x_2+L_3\cdot x_3+L_4\cdot x_4+L_5\cdot x_5}{L_1+L_2+L_3+L_4+L_5}=\frac{2R\sqrt{2}\cdot 2R+R\cdot 0.5R+R\cdot 0+R\pi\cdot\left(-\frac{2R}{\pi}\right)+\frac{R\pi}{2}\left(\frac{2R}{\pi}\right)}{2R\sqrt{2}+R+R+R\pi+\frac{R\pi}{2}}=\frac{5.157}{9.538}R$$

$$x_C=0.54R=5.4cm$$

$$y_C=\frac{L_1\cdot y_1+L_2\cdot y_2+L_3\cdot y_3+L_4\cdot y_4+L_5\cdot y_5}{L_1+L_2+L_3+L_4+L_5}=\frac{2R\sqrt{2}\cdot R+R\cdot 0+R\cdot 0.5R+R\pi\cdot 2R+\frac{R\pi}{2}\left(4R-\frac{2R}{\pi}\right)}{2R\sqrt{2}+R+R+R\pi+\frac{R\pi}{2}}=\frac{14.886}{9.538}R$$

$$y_C=1.5607R=15.607cm$$

2. Obrtna površina koja nastaje obrtanjem oko ξ ose za pun krug.
   Ugao rotacije za pun krug je $\varphi=2\pi$ rad
   $\eta_c=y_c+R=2.5607R=25.607$ udaljenost težišta od ξ ose

$$A_\xi=\varphi L\eta_c=2\pi\cdot 95.38\cdot 25.607=15\,346.024\ cm^2$$

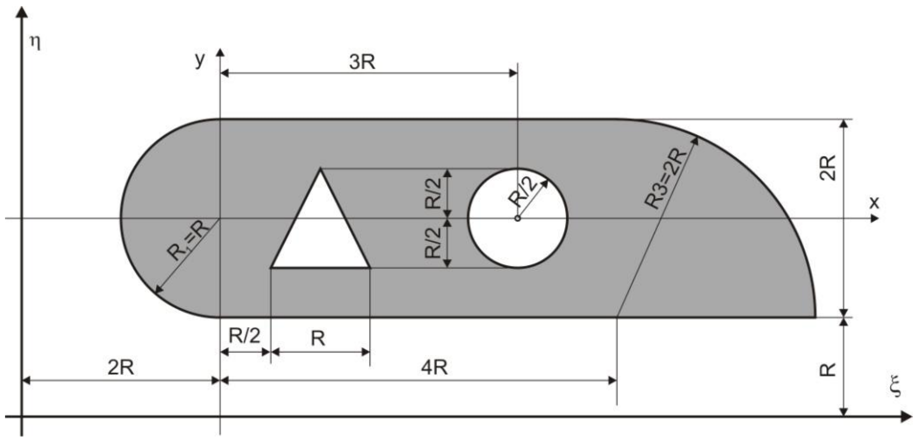
3. Obrtnu površinu koja nastaje obrtanjem oko η ose za 180^{o}.
   Ugao rotacije za polovinu kruga je $\varphi=\pi$ rad
   $\xi_c=x_c+2R=0.5441R+2R=2.54R=25.4$ udaljenost težišta od η ose

Za $\alpha=180°=\pi$ $\quad A_\eta=\varphi L\xi_c=2\pi L\xi_c\frac{\alpha°}{360°}=\pi\cdot 95.38\cdot 25.4=7610.98\ \text{cm2}$

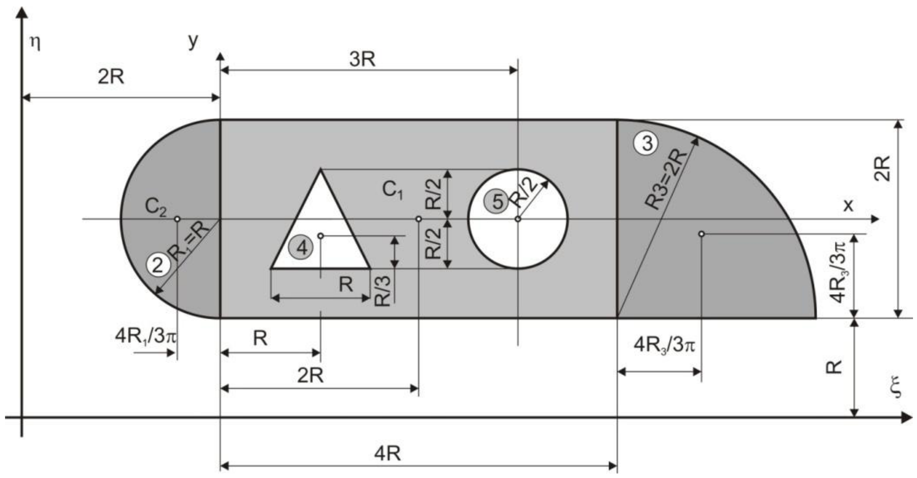
Primer 3.2

Za ravnu površ datu na slici, R=1cm,

1. Odrediti koordinate težišta ravne površi za date koordinatne ose x i y
2. Obrtnu zapreminu koja nastaje obrtanjem oko ξ ose za pun krug.
3. Obrtnu zapreminu koja nastaje obrtanjem oko η ose za 90^o.



Ovu površinu treba rasčlaniti na poznate površine, i definisati površine i koordinate njihovih težišta.



$A_1 = 4R \cdot 2R = 8R^2 = 8\ cm^2 \quad C_1(2R,0) = (2;0)$

$A_2 = \frac{R^2\pi}{2} = 1.57cm^2, \quad C_2\left(-\frac{4R}{3\pi};0\right) = (-0.424;0)$

$A_3 = \frac{(2R)^2\pi}{4} = 3.141cm^2, \quad C_3\left(4R + \frac{4\cdot(2R)}{3\pi}; -\left(R - \frac{4\cdot(2R)}{3\pi}\right)\right) = (4.848;-0.151)$

$A_4 = \frac{R\cdot R}{2} = 0.5R^2 = 0.5\ cm^2 \quad C_4\left(R, -\left(\frac{R}{2} - \frac{R}{3}\right)\right) = (1;-0.166)$

$A_5 = \left(\frac{R}{2}\right)^2 \pi = \frac{R^2\pi}{4} = 0.785cm^2, \quad C_5(3R;0) = (3;0)$

$A = \sum A = A_1 + A_2 + A_3 - A_4 - A_5$

$A = 8R^2 + \frac{R^2\pi}{2} + R^2\pi - \frac{R^2}{2} - \frac{R^2\pi}{4} = \frac{R^2}{4}(32 + 2\pi + 4\pi - 2 - \pi) = \frac{30+5\pi}{4}R^2 = 11.427R^2$

$A = 8 + 1.57 + 3.141 - 0.5 - 0.785 = 11.427\ cm^2$

1. Koordinate težišta za date ose x i y

$$x_c = \frac{\sum A_i \cdot x_i}{\sum A_i} = \frac{A_1 \cdot x_1 + A_2 \cdot x_2 + A_3 \cdot x_3 - A_4 \cdot x_4 + A_5 \cdot x_5}{A_1 + A_2 + A_3 - A_4 + A_5}$$

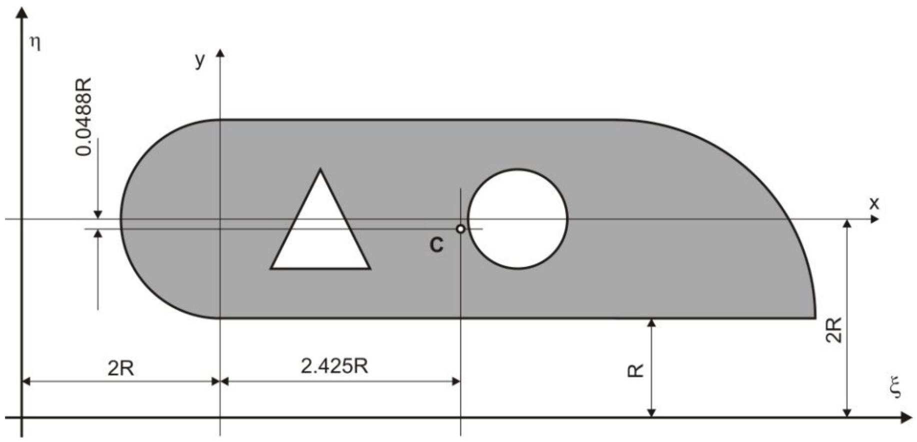
$$x_c = \frac{8R^2 \cdot 2R + \frac{R^2\pi}{2} \cdot \left(-\frac{4R}{3\pi}\right) + R^2\pi \cdot \left(4R + \frac{8R}{3\pi}\right) - \frac{R^2}{2} \cdot R - \frac{R^2\pi}{4} \cdot 3R}{8R^2 + \frac{R^2\pi}{2} + R^2\pi - \frac{R^2}{2} - \frac{R^2\pi}{4}} = \frac{16 - \frac{2}{6} + 4\pi + \frac{8}{3} - \frac{1}{2} - \frac{3\pi}{4}}{\frac{30+5\pi}{4}}$$

$$= \frac{\frac{192 - 4 + 48\pi + 32 - 6 - 9\pi}{12}}{\frac{30+5\pi}{4}} \cdot R = \frac{214 + 39\pi}{90 + 15\pi} \cdot R = 2.425R$$

$$x_c = \frac{8 \cdot 2 + 1.57(-0.424) + 3.141 \cdot 4.848 - 0.5 \cdot 1 - 0.785 \cdot 3}{11.427} = 2.425$$

$$y_c = \frac{8R^2 \cdot 0 + \frac{R^2\pi}{2} \cdot 0 + R^2\pi \cdot \left(-R + \frac{8R}{3\pi}\right) - \frac{R^2}{2} \cdot \left(-\frac{R}{6}\right) - \frac{R^2\pi}{4} \cdot 0}{8R^2 + \frac{R^2\pi}{2} + R^2\pi - \frac{R^2}{2} - \frac{R^2\pi}{4}} = \frac{0 + 0 - \pi + \frac{8}{3} + \frac{1}{12} - 0}{\frac{30+5\pi}{4}} = \frac{-12\pi + 33}{90 + 15\pi} R = -0.0488R$$

$$y_c = \frac{8 \cdot 0 + 1.57 \cdot 0 + 3.14 \cdot (-0.151) - 0.5 \cdot (-0.166) - 0.785 \cdot 0}{11.427} = -0.0488$$



2. Obrtna zapremina koja nastaje obrtanjem oko ξ ose za pun krug.
   Ugao rotacije za pun krug je φ=2π rad
   $\eta_c = y_c + 2R = -0.0488 + 2R = 1.9512R = 1.951cm$ udaljenost težišta od ξ ose

$$V_\xi = \varphi A \eta_c = 2\pi A \eta_c = 2\pi \cdot 11.427 \cdot 1.951 = 140.09\ cm^3$$

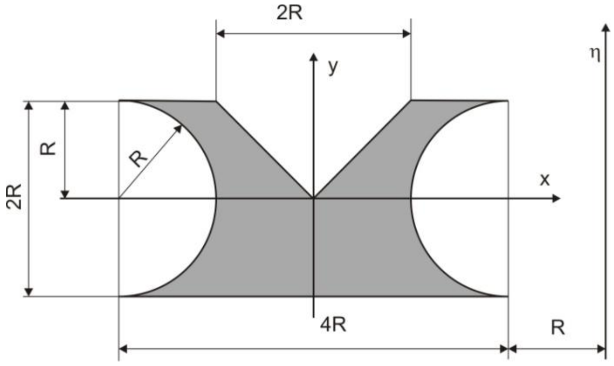
3. Obrtna zapremina koja nastaje obrtanjem oko η ose za 90°.
   Ugao rotacije φ=π/2=90° rad
   $\xi_c = x_c + 2R = 2.425R + 2R = 4.425R = 4.425cm$ udaljenost težišta od η ose

$$V_\eta = \varphi A \xi_c = 2\pi A \xi_c \frac{90°}{360°} = \frac{\pi}{2} \cdot 11.427 \cdot 4.425 = 79.426\ \text{cm}^3$$

Primer 3.3

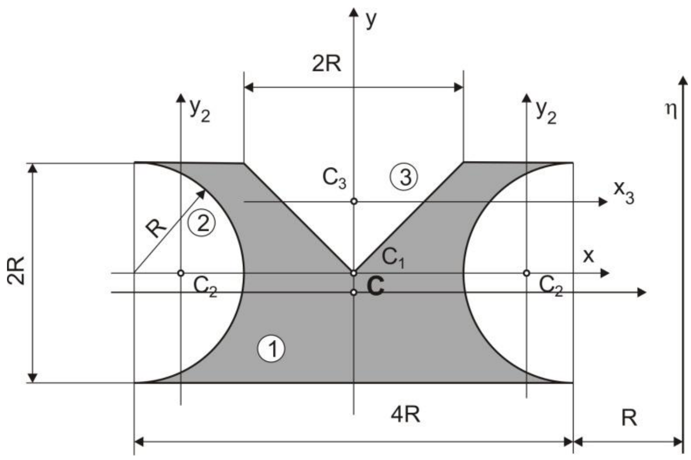
Za ravnu površ datu na slici, R=1cm,

1. Odrediti koordinate težišta ravne površi za date koordinatne ose x i y
2. Obrtnu zapreminu koja nastaje obrtanjem oko η ose za pun krug.



Ako se pogleda slika uočava se simetrija , pa je osa y osa geometrijske simetrije, to je i osa materijalne simetrije jer je površina homogena.

Površinu izdeliti na manje površi za koje su poznate koordinate težišta i površine.



$A_1 = 4R \cdot 2R = 8R^2 = 8cm^2, \quad C_1(0{,}0)$

$A_2 = \frac{R^2\pi}{2} = 25.133cm^2, \quad C_2\left(2R - \frac{4R}{3\pi}; 0\right) = (1.575R; 0) = (6.302; 0)$

$A_3 = \frac{2R \cdot R}{2} = R^2 = 16\ cm^2, \quad C_3\left(0; \frac{2R}{3}\right) = (0; 0.666R) = (0; 2.666)$

$$A = \sum A = A_1 - 2A_2 - A_3 = 8R^2 - 2\frac{R^2\pi}{2} - R^2 = (7-\pi)R^2 = 61.735\ cm^2$$

$$\sum A_i \cdot y_i = A_1 \cdot y_1 - 2A_2 \cdot y_2 - A_3 \cdot y_3 = 8R^2 \cdot 0 - 2\frac{R^2\pi}{2} \cdot 0 - R^2\frac{2R}{3} = -\frac{2R^3}{3} = -42.666cm^2$$

$$y_c = \frac{\sum A_i \cdot y_i}{A} = \frac{A_1 \cdot y_1 - 2A_2 \cdot y_2 - A_3 \cdot y_3}{A_1 - 2A_2 - A_3} = \frac{8R^2 \cdot 0 - 2\frac{R^2\pi}{2} \cdot 0 - R^2\frac{2R}{3} = -\frac{2R^3}{3}}{8R^2 - 2\frac{R^2\pi}{2} - R^2} = \frac{-\frac{2R^3}{3}}{(7-\pi)R^2} = \frac{-2R}{3(7-\pi)} = -0.691cm$$

Koordinate težišta: $x_c = 0,\ y_c = -0.691cm$

Obrtna zapremina:

$$V_\eta = \varphi A \xi_c = 2\pi A \xi_c = 2\pi \cdot (7-\pi)R^2 \cdot 3R = 2\pi \cdot 61.735 \cdot 3 = 1\ 163.677\ \text{cm}^3$$