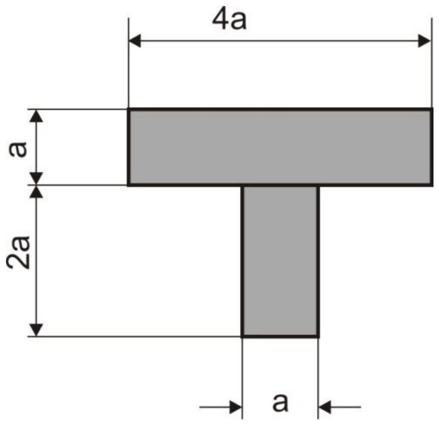
Zadatak 1.

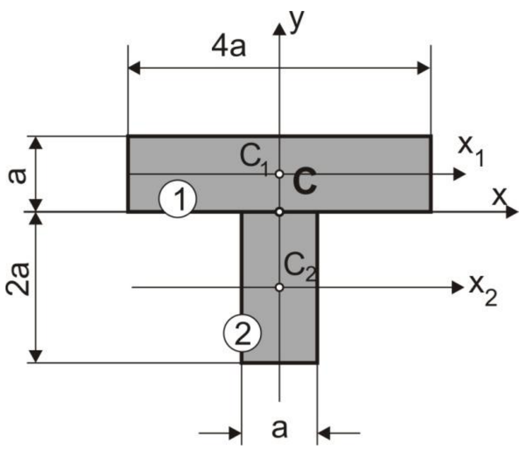


Za datu sliku, a=1cm, odrediti moment inercije i glavne momente inercije kao i poluprečnike elipse inercije:

1. Korišćenjem težišnih momenata inercije za delove slike
2. Korišćenjem ivičnih momenata inercije za delove slike

Rešenje:

Za datu sliku odabrati koordinatni sistem u kome treba odrediti težište ravne površine



Za odabranu ose x i y

$A_1 = 4a \cdot a = 4a^2 = 4cm^2, \quad C_1\left(o, \frac{a}{2}\right) = (0{,}0.5)$

$A_2 = a \cdot 2a = 2a^2 = 2cm^2, \quad C_1(o, -a) = (0, -1)$

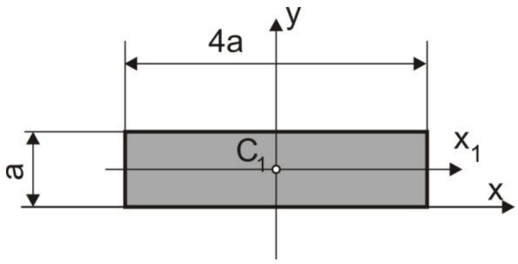
$A = \sum A = A_1 + A_2 = 4a^2 + 2a^2 = 6a^2 = 6\ cm^2$

$S_x = \sum A_i \cdot y_i = A_1 \cdot y_1 + A_2 \cdot y_2 = 4a \cdot \frac{a}{2} + 2a \cdot (-a) = 0$

$S_y = \sum A_i \cdot x_i = A_1 \cdot x_1 + A_2 \cdot x_2 = 4a \cdot 0 + 2a \cdot 0 = 0$

$x_c = \frac{S_y}{A} = 0; \quad y_c = \frac{S_x}{A} = 0;$

Iz tablica izvaditi podatke za odvojene poznate površine:



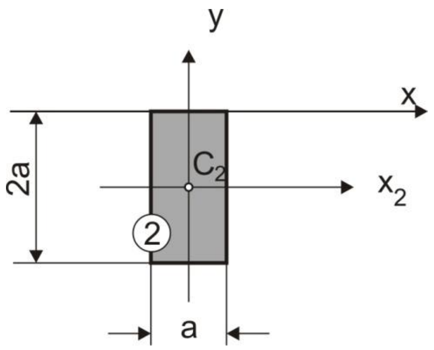
$I_{x1} = \frac{ch^3}{12} = \frac{4a \cdot a^3}{12} = \frac{4a^4}{12} = 0.333cm^4$

$I_y = \frac{c^3 h}{12} = \frac{64a^3 \cdot a}{12} = \frac{64a^4}{12} = \frac{16a^4}{3} = 5.333cm^4$

$I_{x1y} = 0$

Za ivičnu osu ako postoji u tablicama

$I_{x01} = \frac{ch^3}{3} = \frac{4a \cdot a^3}{3} = \frac{4a^4}{3} = 1.3333cm^4$ i $I_{xy01} = 0$



$I_{x2} = \frac{ch^3}{12} = \frac{a \cdot 8a^3}{12} = \frac{8a^4}{12} = \frac{2a^4}{3} = 0.666cm^4$

$I_{y2} = \frac{c^3 h}{12} = \frac{a^3 \cdot 2a}{12} = \frac{2a^4}{12} = \frac{a^4}{6} = 0.1666cm^4$

$I_{x2y} = 0$

Za ivičnu osu ako ima u tablicama

$I_{x02} = \frac{ch^3}{3} = \frac{a \cdot 8a^3}{3} = \frac{8a^4}{3} = 2.666cm^4$ i $I_{xy02} = 0$

1. Korišćenjem težišnih momenata inercije za delove slike

$I_x = I_{x1} + y_1^2 \cdot A_1 + I_{x2} + y_2^2 A_2 = \frac{a^4}{3} + \left(\frac{a}{2}\right)^2 \cdot 4a^2 + \frac{2a^4}{3} + a^2 \cdot 2a^2 = 4a^4 = 4cm^4$

$I_x = 0.3333 + 0.5^2 \cdot 4 + 0.6666 + 1 \cdot 2 = 3.9999\ cm^4$

Y osa je ujedno i težišna osa za obe slike pa nema položajnih momenata i u oba slučaja je:

$I_y = I_{y1} + I_{y2} = \frac{16a^4}{3} + \frac{1a^4}{6} = \frac{33}{6}a^4 = \frac{11}{2}a^4 = 5.3333 + 0.1666 = 5{,}5\ cm^4$

Za centrifugalni moment inercije pošto je y osa osa simetrije je jednak 0 $I_{xy} = 0$

2. Korišćenjem težišnih ivičnih momenata inercije za delove slike

$I_x = I_{x01} + I_{x02} = 1.333 + 2.666 = 3.999\ cm^2$

$I_x = I_{x01} + I_{x02} = \frac{4a^4}{3} + \frac{8a^4}{3} = 4a^4 = 4\ cm^2$

Centrifugalni moment inercije

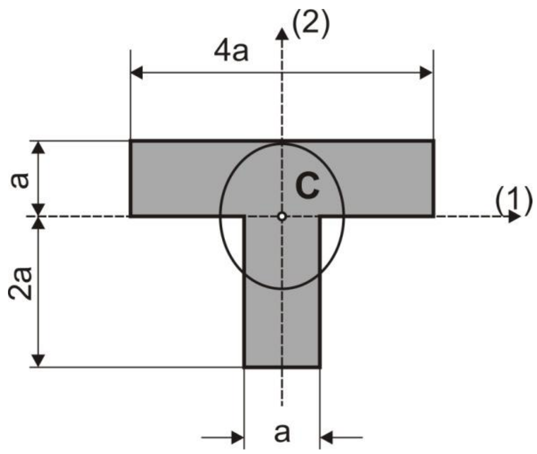
$I_{xy} = I_{xy\,01} + I_{xy\,02} = 0$

Kako je osa y osa simetrijeposmatranih površina ose x i y su glavne težišne ose pa su i momenti veći je I1 a manji I2, odnosno osa x osa (1) a osa y osa (2).

Poluprečnici elipse inercije su:

$i_1 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\frac{11}{2}a^4}{6a^2}} = \sqrt{0.9166a^2} = 0.9574a = 0.9574\ cm$

$i_2 = \sqrt{\frac{I_2}{A}} = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{4a^4}{6a^2}} = \sqrt{0.6666a^2} = 0.81649a = 0.81649\ cm$

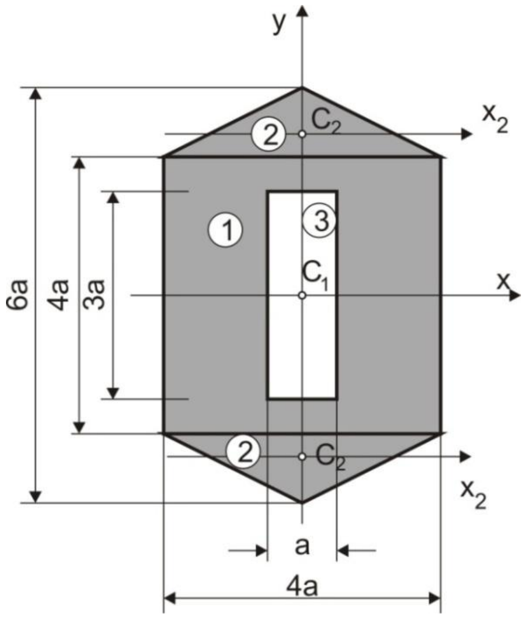


Elipsa inercije zauzima oko jedne trećine površine

Elipsa inercije ne sme da izađe iz preseka

Zadatak 2.

Za datu sliku, a=2cm, odrediti moment inercije i glavne momente inercije kao i poluprečnike elipse inercije.



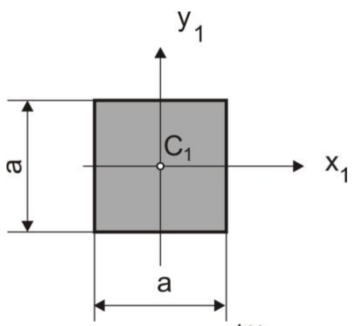
Na osnovu primene simetrije ose x i y su težišne ose kao ose simetrije.

$A_1 = 4a \cdot 4a = 16a^2 = 64cm^2, \quad C_1(0;0)$

$A_2 = \frac{4a \cdot a}{2} = 2a^2 = 8cm^2, \quad C_2\left(0;\frac{11a}{6}\right) = (0;4.666)$

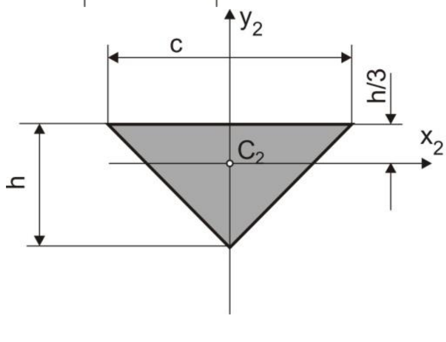
$A_3 = 3a \cdot a = 3a^2 = 12cm^2, \quad C_3(0;0)$

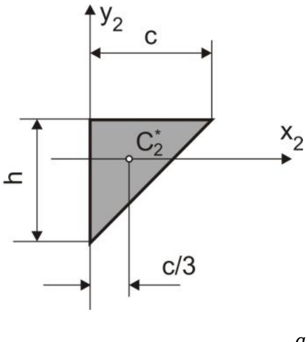
$$A = \sum A = A_1 + 2A_2 - A_3 = 16a^2 + 2 \cdot 2a^2 - 3a^2 = 17a^2 = 68\ cm^2$$



$I_{x1} = I_{y1} = \frac{a^4}{12}$

U zadatku $I_{x1} = I_{y1} = \frac{(4a)^4}{12} = \frac{(4 \cdot 2)^4}{12} = 341.33cm^4$





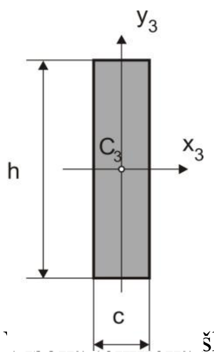
$I_{x2} = \frac{ch^3}{36}$

$I_{y2} = \frac{hc^3}{12}$ za pravougli trougao

U zadatku

$I_{x2} = \frac{4a \cdot a^3}{36} = \frac{4 \cdot 2 \cdot 2^3}{36} = \frac{16}{9} = 1.7777cm^4$

$I_{y2} = 2\frac{a(2a)^3}{12} = 2\frac{2 \cdot 4^3}{12} = 2\frac{128}{12} = 21.333cm^4$



$I_{x3} = \frac{ch^3}{12} = \frac{a \cdot (3a)^3}{12} = \frac{27a^4}{12} = \frac{27 \cdot 2^4}{12} = 36cm^4$

$I_{y3} = \frac{c^3h}{12} = \frac{a^3 \cdot 3a}{12} = \frac{3a^4}{12} = \frac{3 \cdot 2^4}{12} = 4cm^4$

$$I_x = I_{x1} + 2(I_{x2} + y_2^2 \cdot A_2) - I_{x3} = 341.33 + 2(1.777 + 4.666^2 \cdot 8) - 36 = 657.33cm^2$$

$$I_x = \frac{(4a)^4}{12} + 2\left[\frac{4a \cdot a^3}{36} + \left(2a + \frac{a}{3}\right)^2 \frac{4a \cdot a}{2}\right] - \frac{a(3a)^3}{12}$$
$$I_x = \frac{64a^4}{3} + 22a^4 - \frac{9a^4}{4} = \frac{493}{12}a^4 = 657.333cm^4$$

$$I_y = I_{y1} + 2I_{y2} - I_{y3} = 341.333 + 2 \cdot 21.333 - 4 = 379.999 \ cm^4$$

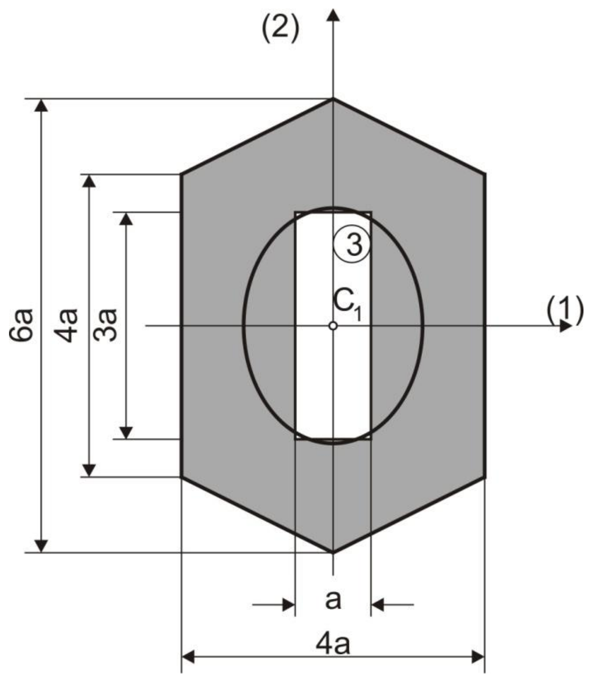
$$I_y = \frac{(4a)^4}{12} + 2\left[2\frac{a \cdot (2a)^3}{12}\right] - \frac{3a \cdot a^3}{12} = \frac{64a^4}{3} + \frac{8a^4}{3} - \frac{1a^4}{4}$$
$$I_y = \frac{285a^4}{12} = \frac{95a^4}{4} = 380cm^4$$

$$I_{xy} = 0$$

$$I_x = I_1; \quad I_y = I_2$$

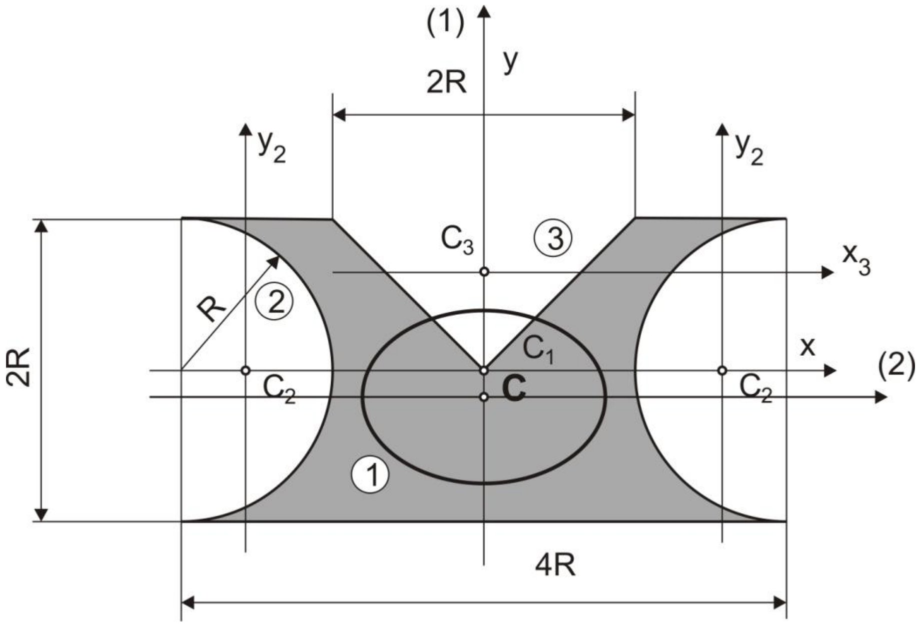
$$i_1 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{492}{12}a^4}{17a^2}} = \sqrt{2.411a^2} = 1.553\,a = 3.11\ cm$$

$$i_2 = \sqrt{\frac{I_2}{A}} = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\frac{95a^4}{4}}{17a^2}} = \sqrt{1.397a^2} = 1.182a = 2.36\ cm$$



Zadatak 3.

Za datu sliku, R=4cm, odrediti moment inercije i glavne momente inercije kao i poluprečnike elipse inercije.



$A_1 = 4R \cdot 2R = 8R^2 = 128cm^2, \quad C_1(0{,}0)$

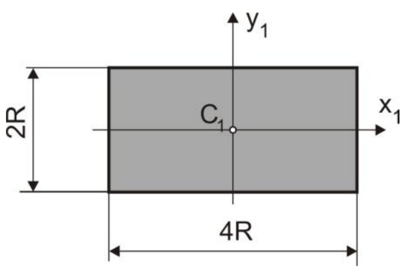
$A_2 = \frac{R^2\pi}{2} = 25.133cm^2, \quad C_2\left(2R - \frac{4R}{3\pi}; 0\right) = (1.575R; 0) = (6.302; 0)$

$A_3 = \frac{2R \cdot R}{2} = R^2 = 16\ cm^2, \quad C_3\left(0; \frac{2R}{3}\right) = (0; 0.666R) = (0; 2.666)$

$$A = \sum A = A_1 - 2A_2 - A_3 = 8R^2 - 2\frac{R^2\pi}{2} - R^2 = (7-\pi)R^2 = 61.735\ cm^2$$

$$S_x = \sum A_i \cdot y_i = A_1 \cdot y_1 - 2A_2 \cdot y_2 - A_3 \cdot y_3 = 8R^2 \cdot 0 - 2\frac{R^2\pi}{2} \cdot 0 - R^2\frac{2R}{3} = -\frac{2R^3}{3} = -42.666cm^2$$

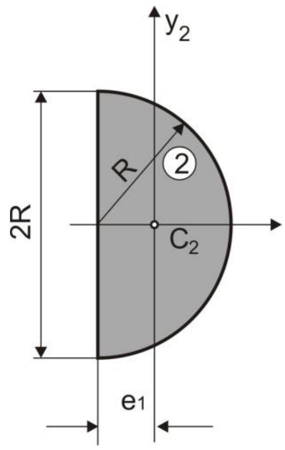
$$y_c = \frac{S_x}{A} = \frac{-\frac{2R^3}{3}}{(7-\pi)R^2} = \frac{-2R^3}{3(7-\pi)R^2} = \frac{-2R}{3(7-\pi)} = -0.691cm$$



$$I_{x1} = \frac{4R(2R)^3}{12} = \frac{32R^4}{12} = \frac{32 \cdot 4^4}{12} = 682.66cm^4$$

$$I_{y1} = \frac{2R(4R)^3}{12} = \frac{128\ R^4}{12} = \frac{32 \cdot 4^4}{3} = 2730.66cm^4$$

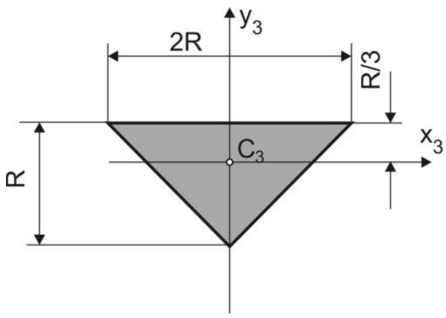
$$I_{xy1} = 0$$

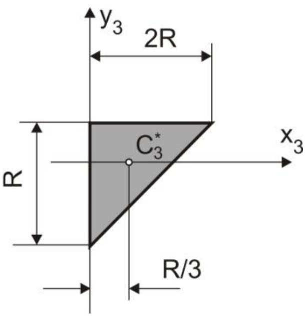


$$e_1 = \frac{4R}{3\pi}$$

$$I_{x2} = \frac{R^4\pi}{8} = \frac{4^4\pi}{8} = \frac{256\pi}{8} = 100.53 \ cm^4$$

$$I_{y2} = R^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 4^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 28.108 \ cm^4$$





$$I_{x3} = \frac{ch^3}{36} = \frac{2R \cdot R^3}{36} = \frac{R^4}{18} = 14.22 \ cm^2$$

$$I_{y31} = \frac{c^3h}{12} = \frac{R^3 \cdot R}{12} = \frac{R^4}{12} = 21.333 \ cm^2$$

$$I_{y3} = 2I_{y31} = 2\frac{c^3h}{12} = \frac{R^3 \cdot R}{6} = \frac{R^4}{6} = 42.66 \ cm^2$$

$$I_x = I_{x1} - 2I_{x2} - (I_{x3} + y_3^2 \cdot A_3) = 682.66 - 2 \cdot 100.53 - (14.22 + 2.66^2 \cdot 16) = 354.17cm^2$$

$$I_x = \frac{4R(2R)^3}{12} - 2\frac{R^4\pi}{8} - \left[\frac{2R \cdot R^3}{36} + \left(\frac{2R}{3}\right)^2 R^2\right]$$

$$I_x = \frac{32R^4}{12} - 2\frac{R^4\pi}{8} - \frac{2+16}{36}R^4 = (96 - 9\pi - 18)\frac{R^4}{36} = (78 - 9\pi)\frac{R^4}{36} = 1.31812R^4 = 353.60cm^4$$

$$I_y = I_{y1} - 2\left(I_{y2} + x_2^2 \cdot A_2\right) - I_{y3} = 2730.66 - 2[28.108 + 6.302^2 \cdot 25.13] - 42.66 = 635.39cm^4$$

$$I_y = \frac{2R(4R)^3}{12} - 2\left[\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)R^4 + \left(2R - \frac{4R}{3\pi}\right)^2 \frac{R^2\pi}{2}\right] - 2\frac{R^4 \cdot R}{12}$$

$$I_y = \frac{32R^4}{3} - \frac{\pi}{4}R^4 + \frac{16}{9\pi}R^4 - 2\left[\left(4R^2 - \frac{16R^2}{3\pi} + \frac{16R^2}{9\pi^2}\right)\frac{R^2\pi}{2}\right] - \frac{R^4}{6}$$

$$I_y = \frac{32R^4}{3} - \frac{\pi}{4}R^4 + \frac{16}{9\pi}R^4 - 4R^4\pi + \frac{16R^4}{3} - \frac{16R^4}{9\pi} - \frac{R^4}{6} = \frac{R^4}{12}\left[128 - 3\pi - 48\pi + 64 - 2\right] =$$

$I_y = \frac{R^4(190-51\pi)}{12} = 2.48157R^4 = 635.28cm^4$

$I_{xy} = 0$

$I_1 = I_y = 635.28\ cm^4$

Kako je x osa pomerena u odnosu na glavnu težišnu osu za yc=0.691 cm to se određuje

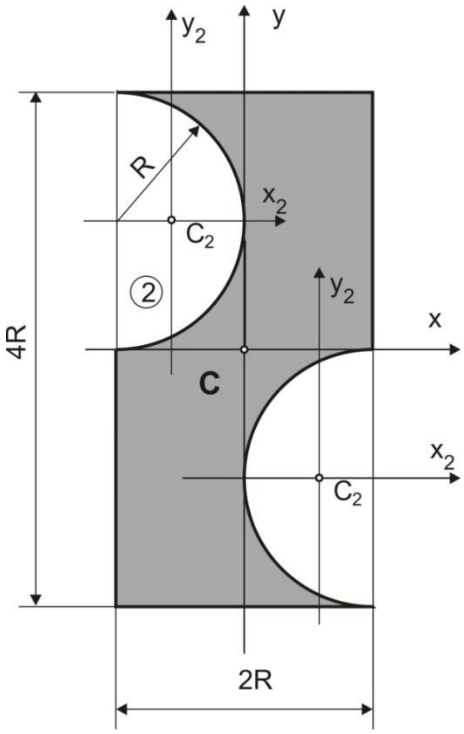
$I_2 = I_x - y_c^2 A = 358.60 - (0.691)^2 \cdot 61.75 = 324.104\ cm^4$

$i_1 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{635.28}{61.75}} = \sqrt{10.287} = 3.21\ cm$

$i_2 = \sqrt{\frac{I_2}{A}} = \sqrt{\frac{324.194}{61.75}} = \sqrt{5.25} = 2.291\ cm$

Zadatak 4.

Za datu sliku, R=3cm, odrediti moment inercije i glavne momente inercije kao i poluprečnike elipse inercije.



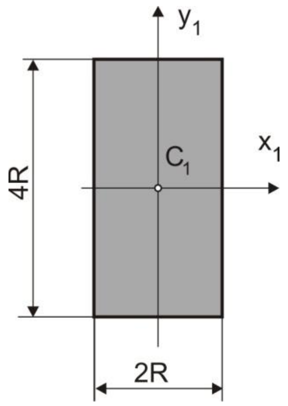
$$A_1 = 4R \cdot 2R = 8R^2 = 72cm^2, \quad C_1(0,0)$$

$$A_2 = \frac{R^2\pi}{2} = 14.137cm^2,$$

$$C_2\left(R - \frac{4R}{3\pi}; -R\right) = (0.575R, -R) = (1.727; 3)$$

$$A = \sum A = A_1 - 2A_2 = 8R^2 - 2\frac{R^2\pi}{2} = (8-\pi)R^2$$

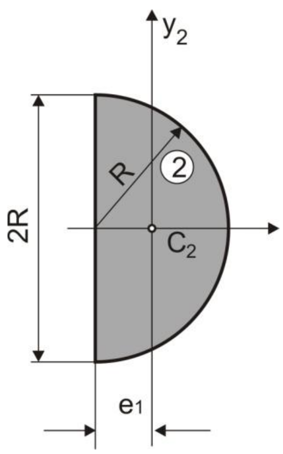
$$A = 43.725 \ cm^2$$



$$I_{x1} = \frac{2R(4R)^3}{12} = \frac{32R^4}{3} = \frac{32 \cdot 3^4}{3} = 864 \ cm^4$$

$$I_{y1} = \frac{4R(2R)^3}{12} = \frac{32\ R^4}{12} = \frac{32 \cdot 3^4}{12} = 216\ cm^4$$

$$I_{x1y1} = 0$$



$$e_1 = \frac{4R}{3\pi}$$

$$I_{x2} = \frac{R^4\pi}{8} = \frac{3^4\pi}{8} = \frac{81\ \pi}{8} = 31.808 \ cm^4$$

$$I_{y2} = R^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 3^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 8.894 \ cm^4$$

$$I_{x2y2} = 0$$

$$I_x = I_{x1} - 2(I_{x2} + y_2^2 \cdot A_2) = \frac{32R^4}{3} - 2\left[\frac{81\ \pi}{8} + R^2 \cdot \frac{R^2\pi}{2}\right] =$$

$$I_x = 864 - 2(31.808 + 3^2 \cdot 14.137) = 545.918cm^2$$

$$I_y = I_{y1} - 2\left(I_{y2} + x_2^2 \cdot A_2\right) = \frac{32R^4}{12} - 2\left[R^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) + \left(R - \frac{4R}{3\pi}\right)^2 \cdot \frac{R^2\pi}{2}\right] =$$

$I_y = 216 - 2(8.894 + 1.727^2 \cdot 14.137) = 113.883\ cm^4$

$I_{xy} = I_{x1y1} - 2(I_{x1y2} + x_2 \cdot y_2.A_2) = 0 - 2\left[0 + (-R) \cdot \left(R - \frac{4R}{3\pi}\right) \cdot \frac{R^2\pi}{2}\right] = 2(3 \cdot 1.727 \cdot 14.137) =$

$I_{xy} = 146.487 cm^4$

$\tan 2\alpha = \frac{-2I_{xy}}{I_x - I_y} = \frac{-2(-146.487)}{545.918 - 113.883} = -0.67812 \rightarrow 2\alpha = -34.142° \rightarrow \alpha = -17.071°$

$I_{12} = \frac{1}{2}(I_x + I_y) \pm \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2} =$

$I_{12} = \frac{1}{2}(545.918 + 113.883) \pm \frac{1}{2}\sqrt{(545.918 - 113.883)^2 + 4 \cdot 146.487^2} =$

$I_{12} = \frac{1}{2}(659.801) \pm \frac{1}{2}\sqrt{186697.44 + 85833.76} = 329.900 \pm 261.022$

$I_1 = 590.922 cm^4$

$I_2 = 68.878 cm^4$

$i_2 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{590.922}{43.725}} = \sqrt{13.145} = 3.676\ cm$

$i_2 = \sqrt{\frac{I_2}{A}} = \sqrt{\frac{68.878}{43.725}} = \sqrt{1.575} = 1.255\ cm$